Section 7.3: Compound Events

Because we are using the framework of set theory to analyze probability, we can use unions, intersections and complements to break complex events into compositions of events for which it may be easier to calculate probabilities. An event that can be described in terms of the union, intersection or complement of events is called **a compound event**. We get formulas for the probabilities of compound events similar to the counting formulas we encountered earlier. in fact, we can derive these directly from our counting formulas for equally likely sample spaces.

Let E and F be events in a sample space, then

- $E \cup F$(the **union** of E and F) is the event consisting of those outcomes which are in at least one of the two events, that is those outcomes which are in either E or F or both.
- $E \cap F$ (the **intersection** of E and F) is the event consisting of those outcomes which are in both of the events E and F.
- The event E' (the **complement** of E) is the event consisting of those outcomes which are not in E.

Example * Roll **a pair of fair six-sided dice**, one red and one green, and observe the pair of numbers on the two uppermost faces. The sample space for this experiment shown below is a list of the possible pairs of numbers, listing the number on the red die first and the number on the green die second.

$$\begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

(a) Let E be the event: "The sum of the pair of numbers on the uppermost faces is 7". List the elements of E.

$\{(6,1),(5,2),(4,3),(3,4),(2,5),(1,6)\}$

(b) Let F be the event: "At least one of the numbers on the uppermost faces is 6". How many outcomes are in the event F?

$\{(6,1),(6,2),(6,3),(6,4),(6,5),(6,6),(5,6),(4,6),(3,6),(2,6),(1,6)\}$

so there are 11 possible outcomes in F.

(c) List the outcomes in $E \cap F$.

$\{(6,1),(1,6)\}$

(d) How many outcomes are in the event $E \cup F$.

Notice I was not asked to list them so I can compute as follows:

$n(E \cup F) = n(E) + n(F) - n(E \cap F) = 6 + 11 - 2 = 15$

(e) How many outcomes are in the event E'.

$n(E') = n(U) - n(E) = 36 - 6 = 30$

Example An experiment with outcomes a, b, c, d is described by the probability table

Outcomes	Probability
a	0.2
b	0.1
c	0.15
d	0.55

Consider the events $E = \{a, b, c\}$ and $F = \{b, c, d\}$. What is $\mathrm{P}(E \cap F)$?

$E \cap F = \{b, c\}$ so $\mathrm{P}(E \cap F) = \mathrm{P}(\{b, c\}) = \mathrm{P}(b) + \mathrm{P}(c) = 0.1 + 0.15 = 0.25$.

Complement Rule Last time, we saw that if E is an event in a sample space with equally likely outcomes, we know that

$$\boxed{\mathrm{P}(E') = 1 - \mathrm{P}(E).}$$

This rule is **true for any sample space** and is not difficult to prove. We can also derive the following formula for the probability of the union of two event, which we will demonstrate in the case where the sample space has equally likely outcomes:
Let E and F be events in a sample space S, then

$$\boxed{\mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F).}$$

If E and F are events in a sample space with equally likely outcomes, then

$$\mathrm{P}(E \cup F) = \frac{n(E \cup F)}{n(S)} = \frac{n(E) + n(F) - n(E \cap F)}{n(S)} = \frac{n(E)}{n(S)} + \frac{n(F)}{n(S)} - \frac{n(E \cap F)}{n(S)} = \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F).$$

Example * We can see this in action in our previous example: We show the event E in red below, we show the event F in blue, we show the event $E \cap F$ in green and the event $E \cup F$ in magenta.

$$\left\{\begin{array}{llllll} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}\right\} \qquad \left\{\begin{array}{llllll} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}\right\}$$

$$\left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\} \qquad \left\{\begin{matrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{matrix}\right\}$$

(a) Verify that $P(E \cup F) = P(E) + P(F) - P(E \cap F)$.

(b) Use the formulas given above to find $P(E' \cup F)$.

$$P(E' \cup F) = \frac{n(E' \cup F)}{36} = \frac{n(E')}{36} + \frac{n(F)}{36} - \frac{n(E' \cap F)}{36} = \frac{n(E') + n(F) - n(E' \cap F)}{36} = \frac{30 + 11 - 9}{36} = \frac{32}{36}$$

For this next set of examples E and F are events in some unknown sample space S.

Example Let E and F be events in a sample space S. If $P(E) = .3$, $P(F) = .8$ and $P(E \cap F) = .2$, what is $P(E \cup F)$?

$$P(E \cup F) = P(E) + P(F) - P(E \cap F) = 0.3 + 0.8 - 0.2 = 0.9$$

Example Let A and B be two events. If $P(A \cup B) = 0.7$, $P(A) = 0.3$, and $P(B) = 0.4$ then $P(A \cap B) =$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$
$$0.7 = 0.3 + 0.4 - P(A \cap B)$$

so $P(A \cap B) = 0$. In other words $A \cap B$ never happens which means $A \cap B$ is the empty set so A and B are disjoint.

Example Let E and F be events in a sample space, S. If $P(E) = .5$, $P(F') = .4$ and $P(E \cup F) = .9$, find $P(E \cap F)$.

If we knew $\mathrm{P}(F)$ the calculation would be straightforward. But $\mathrm{P}(F) = 1 - \mathrm{P}(F') = 1 - 0.4 = 0.6$.

$$\begin{aligned} \mathrm{P}(E \cup F) &= \mathrm{P}(E) + \mathrm{P}(F) - \mathrm{P}(E \cap F) \\ 0.9 &= 0.5 + 0.6 - \mathrm{P}(E \cap F) \end{aligned}$$

so $\mathrm{P}(E \cap F) = 0.2$.

Example At the Bad Donkey Stables, 50% of the donkeys bite, 40% kick, and 20% do both. You are helping out for the day and you choose a donkey at random to groom. What is the probability that the donkey you choose will either bite or kick (or both)?

Let B be the subset (= event) of donkeys that bite and let K be the subset of donkeys that kick. We are told $\mathrm{P}(B) = 0.5$, $\mathrm{P}(K) = 0.4$ and $\mathrm{P}(B \cap K) = 0.2$. We have been asked for $\mathrm{P}(B \cup K) = \mathrm{P}(B) + \mathrm{P}(K) - \mathrm{P}(B \cap K) = 0.5 + 0.4 - 0.2 = 0.7$ or 70%.

Example there are 10,000 undergraduate students currently enrolled at the University of Notthe Same. Two thousand of the undergraduate students are currently enrolled in a math class, two thousand, five hundred are currently enrolled in an English class and five hundred are currently enrolled in both and English and a math class. If you choose a student at random from among undergraduates currently enrolled at the University of Notthe Same,

(a) what is the probability that they are enrolled in either an English class or a math class or both?

Let M be the subset of students enrolled in a math class and let E be the subset enrolled in an English class. The sample space S is all undergraduate students. We are told $\mathrm{P}(M) = \dfrac{n(M)}{n(S)} = \dfrac{2,000}{10,000}$, $\mathrm{P}(E) = \dfrac{n(E)}{n(S)} = \dfrac{2,500}{10,000}$ and $\mathrm{P}(E \cap M) = \dfrac{n(E \cap M)}{n(S)} = \dfrac{500}{10,000}$. We are asked for $\mathrm{P}(E \cup M) = \dfrac{n(E \cup M)}{n(S)} = \dfrac{2,500}{10,000} + \dfrac{2,000}{10,000} - \dfrac{500}{10,000} = \dfrac{4,000}{10,000} = 0.4$ or 40%.

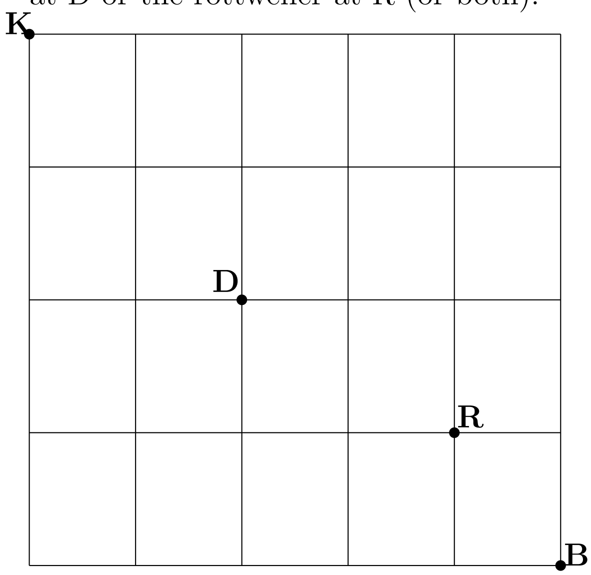
(b) what is the probability that the student is not enrolled in either an English or a math class?

Here we are being asked for $\mathrm{P}\big((E \cup M)'\big) = 1 - \mathrm{P}(E \cup M) = 1 - 0.4 = 0.6$.

Example In the town of Novax, the probability that a child will contract measles is 0.45, the probability that the child will contract whooping cough is 0.6 and the probability that the child will contract both is 0.3, what is the probability that the child will contract at least one of these diseases?

Let M denote the subset of children with measles and let W denote the subset of children with whooping cough. Then we are being told $\mathrm{P}(M) = 0.45$ $\mathrm{P}(W) = 0.6$ and $\mathrm{P}(M \cap W) = 0.3$. We are being asked for $\mathrm{P}(M \cup W) = \mathrm{P}(M) + \mathrm{P}(W) - \mathrm{P}(M \cap W) = 0.45 + 0.6 - 0.3 = 0.75$.

Example Kristina randomly chooses a route from K to B (see map below) with no backtracking for her morning run. What is the probability that the route she chooses will take her past either the doberman at D or the rottweiler at R (or both).



Here the sample space S is the set of all routes with no backtracking. As we have argued before $n(S) = C(5+4,5) = C(9,5) = 126$. Let D be the subset of routes which take her past the doberman and R the subset of routes that take her past the rottweiler. We have counted each of these in the past.

$n(D) = C(2+2,2) \cdot C(3+2,3) = C(4,2) \cdot (5,3) = 6 \cdot 10 = 60$

and

$n(R) = C(4+3,3) \cdot C(1+1,1) = C(7,3) \cdot (2,1) = 35 \cdot 12 = 70$

We also know how to compute $n(D \cap R)$: it is the set of all routes that go through both D and R so

$n(D \cap R) = C(2+2,2) \cdot C(2+1,2) \cdot C(1+1,1) = C(4,2) \cdot (3,2) \cdot C(2,1) = 6 \cdot 3 \cdot 2 = 36.$

We are being asked for $\mathrm{P}(D \cup R) = \dfrac{n(D \cup R)}{n(S)} = \dfrac{n(D)}{n(S)} + \dfrac{n(R)}{n(S)} - \dfrac{n(D \cup R)}{n(S)} = \dfrac{60}{126} + \dfrac{70}{126} - \dfrac{36}{126} = \dfrac{94}{126}$.

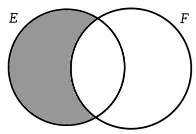
Venn Diagrams

We can use Venn diagrams to represent probabilities by recording the appropriate probability in each basic region. Note the sum of all probabilities in the sample space(represented by the outer rectangle) is one.

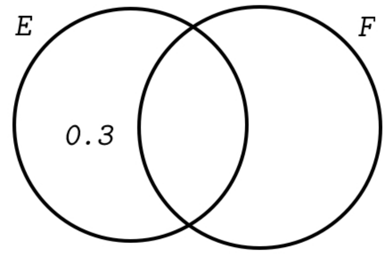
Example If $\mathrm{P}(E) = .4$, $\mathrm{P}(F) = .5$ and $\mathrm{P}(E \cap F') = .3$,

(a) Use a Venn diagram to find $\mathrm{P}(E \cap F)$.

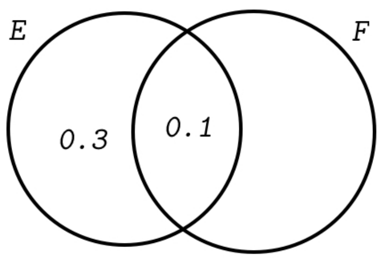
To use a Venn diagram for probabilities we proceed just as we did for counting. First we need the value for a simple piece of the diagram. In this case $E \cap F'$ is the shaded simple piece:



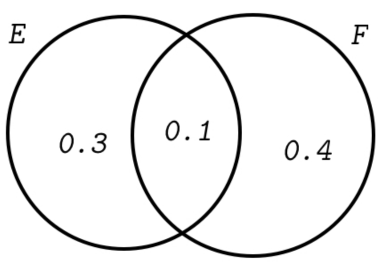
Hence we start by writing



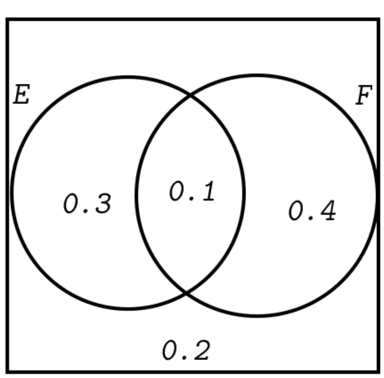
Since $\mathrm{P}(E) = 0.4$ we need to write 0.1 in $E \cap F$ to get the probabilities to add up correctly.



Since $\mathrm{P}(F) = 0.5$ we need to put 0.4 in the remaining piece of F.



Finally, since the total of all the probabilities must be 1, we need to put 0.2 in the outer region.



Then $\mathrm{P}(E \cap F) = 0.1$.

(b) Find $\mathrm{P}(E \cup F)$.

$\mathrm{P}(E \cup F) = 0.3 + 0.1 + 0.4 = 0.8.$

Note If $\mathrm{P}(E) = 1$, it means that E is a **certain** event and if $\mathrm{P}(E) = 0$, the event is impossible.

We say two events E and F are **mutually exclusive** if they have no outcomes in common. This is equivalent to any one of the following conditions:

- E and F are disjoint,
- $E \cap F = \emptyset$,
- $\mathrm{P}(E \cap F) = 0$.

When two events E and F are mutually exclusive, we see from the formula for $\mathrm{P}(E \cup F)$ above that can calculate $\mathrm{P}(E \cup F)$ by adding probabilities, that is

$$\boxed{E \text{ and } F \textbf{ mutually exclusive} \text{ implies that } \quad \mathrm{P}(E \cup F) = \mathrm{P}(E) + \mathrm{P}(F).}$$

Example Two fair six sided dice are rolled, and the numbers on their top faces are recorded. Consider the following events:

E : both numbers are odd

F : the sum of the two numbers is odd

G : at least one of the numbers is a 5

$$\begin{array}{cccccc} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{array}$$

Which of the following statements about these events is true?

(a) E and G are mutually exclusive

(b) E and F are mutually exclusive

(c) F and G are mutually exclusive

(a) $(5,5)$ is a simple event in both E and G so they are not mutually exclusive.
(b) If both numbers are odd, then the sum is even, so E and F are mutually exclusive.
(b) $(5,2)$ is a simple event in both F and G so they are not mutually exclusive.

Example if we draw 5 cards from a deck of 52 at random, what is the probability that the hand of cards we draw will have either 4 aces or 4 kings?

The sample space S is the set of all 5 card subsets of the 52 cards so $n(S) = C(52, 5)$. The subset of hands with 4 aces A is $C(4, 4)\cdot C(48, 1) = 48$. The subset of hands with 4 kings K is $C(4, 4)\cdot C(48, 1) = 48$. The subsets A and K are disjoint so $n(A\cup K) = 96$. Hence the answer is $\mathrm{P}(A\cup K) = \dfrac{n(A\cup K)}{n(S)} = \dfrac{96}{2,598,960} = 3.69378520639025\cdot 10^{-5} =$ 0.0000369378520639025.

Example If we flip a coin 20 times and observe the resulting ordered sequence of heads and tails, (write the answers using combinations of powers, permutations, combinations or factorials as appropriate)

(a) what is the probability that we get either exactly 0 heads or exactly 1 head or 2 heads in the resulting sequence?

The sample space S has size $n(S) = 2^{20}$. Let H_i be the subset of S with i heads, so as we have seen, $n(H_i) = C(20, i)$. We want $n(H_0 \cup H_1 \cup H_2)$ and these three subsets are mutually exclusive so $n(H_0 \cup H_1 \cup H_2) = n(H_0) + n(H_1) + n(H_2) = C(20, 0) + C(20, 1) + C(20, 2) = 1 + 20 + 190 = 211$. Hence the probability is $\mathrm{P}(H_0 \cup H_1 \cup H_2) = \dfrac{C(20, 0) + C(20, 1) + C(20, 2)}{2^{20}}$.

(b) what is the probability that we get at least three heads?

The answer is $\mathrm{P}(H_3 \cup H_4 \cup \cdots \cup H_{20})$. It is easier to compute the complement, $\mathrm{P}(H_0 \cup H_1 \cup H_2)$ since we just did that in (a). Hence the answer is
$\mathrm{P}(H_3 \cup H_4 \cup \cdots \cup H_{20}) = 1 - \dfrac{C(20, 0) + C(20, 1) + C(20, 2)}{2^{20}}$